Theoretical Computer Science 411 (2010) 3287–3292

Contents lists available at ScienceDirect

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcs

Note

$L(2, 1)$-Labelings on the composition of n graphs

Zhendong Shao *, Roberto Solis-Oba [1]

Department of Computer Science, The University of Western Ontario, London, ON, Canada

ARTICLE INFO

Article history:
Received 1 August 2009
Received in revised form 22 February 2010
Accepted 2 March 2010
Communicated by D.-Z. Du

Keywords:
Channel assignment
$L(2, 1)$-labeling
Graph composition

ABSTRACT

An $L(2, 1)$-labeling of a graph G is a function f from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(x) - f(y)| \geq 2$ if $d(x, y) = 1$ and $|f(x) - f(y)| \geq 1$ if $d(x, y) = 2$, where $d(x, y)$ denotes the distance between x and y in G. The $L(2, 1)$-labeling number $\lambda(G)$ of G is the smallest number k such that G has an $L(2, 1)$-labeling with $\max\{f(v) : v \in V(G)\} = k$. Griggs and Yeh conjectured that $\lambda(G) \leq \Delta^2$ for any simple graph with maximum degree $\Delta \geq 2$. In this article, a problem in the proof of a theorem in Shao and Yeh (2005) [19] is addressed and the graph formed by the composition of n graphs is studied. We obtain bounds for the $L(2, 1)$-labeling number for graphs of this type that is much better than what Griggs and Yeh conjectured for general graphs. As a corollary, the present bound is better than the result of Shiu et al. (2008) [21] for the composition of two graphs $G_1[G_2]$ if $\nu_2 < \Delta_2^2 + 1$, where ν_2 and Δ_2 are the number of vertices and maximum degree of G_2 respectively.

1. Introduction

The frequency assignment problem is to assign frequencies to a given group of radio transmitters so that interfering transmitters are assigned frequencies with at least a minimum allowed separation. This problem was formulated as a graph vertex coloring problem by Hale [10]. In a private communication with Griggs, Roberts proposed a variation of the channel assignment problem in which "close" transmitters must receive different channels and "very close" transmitters must receive non-adjacent channels. To translate the problem into the language of graph theory, the transmitters are represented by the vertices of a graph; two vertices are "very close" if they are adjacent and "close" if they are at distance 2 in the graph. Motivated by this problem, Griggs and Yeh [9] proposed the following labeling on a simple graph: An $L(2, 1)$-*labeling* of a graph G is a function f from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(x) - f(y)| \geq 2$ if $d(x, y) = 1$ and $|f(x) - f(y)| \geq 1$ if $d(x, y) = 2$, where $d(x, y)$ denotes the distance between x and y in G. A k-$L(2, 1)$-*labeling* is an $L(2, 1)$-labeling such that no label is greater than k. The $L(2, 1)$-*labeling number* of G, denoted by $\lambda(G)$, is the smallest number k such that G has a k-$L(2, 1)$-labeling.

A large number of articles have been published and devoted to the study of the frequency assignment problem and its connections to graph labelings, in particular to the class of $L(2, 1)$-labelings and its generalizations (see [1–9,11–22]). Over 100 references on the subject are provided in a very comprehensive survey [2]. Due to the inherent hardness of the class of coloring problems, most of these papers considered the computation of λ only for particular classes of graphs.

From the algorithmic point of view it is not surprising that it is NP-complete to decide whether a given graph G allows an $L(2, 1)$-labeling [9]. Hence good lower and upper bounds for λ are clearly welcome. For instance, if G is a diameter 2 graph, then $\lambda(G) \leq \Delta^2$ where Δ is the maximum degree of G. This upper bound is attainable by *Moore graphs*, diameter 2 graphs with $\Delta^2 + 1$ vertices—see [9]; such graphs exist for $\Delta = 2, 3, 7$, and possibly 57.

* Corresponding author. Tel.: +1 519 694 0456.
E-mail addresses: zshao22@csd.uwo.ca, zhdshao0026@126.com (Z. Shao), solis@csd.uwo.ca (R. Solis-Oba).

1 Research partially supported by the Natural Sciences and Engineering Research Council of Canada, grant 227829-2009.

0304-3975/$ – see front matter
doi:10.1016/j.tcs.2010.03.013

The above considerations motivated Griggs and Yeh [9] to conjecture that for any graph G with maximum degree $\Delta \geq 2$ the best upper bound on $\lambda(G)$ is Δ^2. (Note that this is not true for $\Delta = 1$. For example, $\Delta(K_2) = 1$ but $\lambda(K_2) = 2$.) However, papers regarding the upper bound on general graphs are rare. Griggs and Yeh [9] first proved that $\lambda \leq \Delta^2 + 2\Delta$ for general graphs with maximum degree Δ. Chang and Kuo [4] improved the bound to $\Delta^2 + \Delta$, while Král' and Škrekovski [12] further reduced the bound to $\Delta^2 + \Delta - 1$.

Graph products play an important role in network applications. In [19] the Cartesian product and the composition of two graphs were studied and it was proven that the $L(2, 1)$-labeling number of these graphs is bounded above by the square of the maximum degree; unfortunately, the proof for the bound on the $L(2, 1)$-labeling number of the composition of graphs had a mistake, so the bound is only valid for graphs with no isolated vertices. Recently, Shiu et al. [21] presented a new approach for the analysis of the adjacency matrices of certain classes of graphs and derived upper bounds for their $L(2, 1)$-labeling numbers. In this work we address the problem with the proof in [19] and study the $L(2, 1)$-labeling number of the composition of n graphs. We show that the $L(2, 1)$-labelling for the composition of n graphs is much smaller than the square of the maximum degree. As a corollary, the present bound is better than the result of [21] for the composition of two graphs $G_1[G_2]$ if $\nu_2 < \Delta_2^2 + 1$, where ν_2 and Δ_2 are the number of vertices and maximum degree of G_2 respectively.

2. A labeling algorithm

A subset X of $V(G)$ is called an *i-stable set* (or *i-independent set*) if the distance between any two vertices in X is greater than i. A 1-stable set is a usual independent set. A *maximal* i-stable subset X of a set Y of vertices is an i-stable subset of Y such that X is not a proper subset of any other i-stable subset of Y.

Chang and Kuo [4] proposed the following algorithm for computing an $L(2, 1)$-labeling on a given graph.

Algorithm Label(G)
Input: A graph $G = (V, E)$.
Output: The value k of the maximum label.
Idea: In each step, find a maximal 2-stable set from the unlabeled vertices that are at distance at least 2 from the vertices labeled in the previous step. Label all vertices in the 2-stable set with the index i of the current step. The index i starts from 0 and increases by 1 at each step. The maximum label k is the final value of i.
Initialization: Set $X_{-1} = \emptyset$; $V = V(G)$; $i = 0$.
Iteration:

1. Determine Y_i and X_i.
   - **If** $X_{i-1} \neq \emptyset$ **then** set $Y_i = \{x \in V : x \text{ is unlabeled and } d(x, y) \geq 2 \text{ for all } y \in X_{i-1}\}$.
     else Set $Y_i = V$.
   - **If** $Y_i \neq \emptyset$ **then** compute X_i, a maximal 2-stable subset of Y_i; **else** set $X_i = \emptyset$.
2. Label the vertices in X_i (if there are any) with i.
3. $V \leftarrow V \backslash X_i$.
4. If $V \neq \emptyset$ then set $i \leftarrow i + 1$ and go to Step 1.
5. Record the current value of i as k (which is the maximum label). Stop.

Note that the value k computed by the above algorithm is an upper bound on $\lambda(G)$. We would like to find a bound for k in terms of the maximum degree $\Delta(G)$ of G, analogous to existing bounds for the chromatic number $\chi(G)$ in terms of $\Delta(G)$.

Let x be a vertex with the largest label k assigned by Algorithm Label. Define:

- $I_1 = \{i : 0 \leq i \leq k - 1 \text{ and } d(x, y) = 1 \text{ for some } y \in X_i\}$. This is the set of labels of the neighbors of x.
- $I_2 = \{i : 0 \leq i \leq k - 1 \text{ and } d(x, y) \leq 2 \text{ for some } y \in X_i\}$. The set consists of the labels of the vertices at distance at most 2 from x.
- $I_3 = \{i : 0 \leq i \leq k - 1 \text{ and } d(x, y) \geq 3 \text{ for all } y \in X_i\}$. The set consists of the labels not used by vertices at distance at most 2 from x.

It is clear that $|I_2| + |I_3| = k$. For any $i \in I_3$, $x \notin Y_i$ since otherwise $X_i \cup \{x\}$ would be a 2-stable subset of Y_i, which contradicts the choice of X_i. That is, $d(x, y) = 1$ for some vertex y in X_{i-1}; i.e., $i - 1 \in I_1$. Since for every $i \in I_3$, $i - 1 \in I_1$, then $|I_3| \leq |I_1|$. Hence $k = |I_2| + |I_3| \leq |I_2| + |I_1|$.

In order to upper bound k, we will just find a bound for

$$B = |I_1| + |I_2| \tag{1}$$

in terms of $\Delta(G)$.

3. The combinatorial analysis approach

The *composition* of two graphs G and H is the graph $G[H]$ with vertex set $V(G) \times V(H)$, in which (u, v) is adjacent to (u', v') if and only if either $uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$. (See Fig. 1 for an example.)

By the definition of $G[H]$, if $\Delta(G) = 0$, then $G[H]$ consists of disjoint copies of H. Thus $\lambda(G[H]) = \lambda(H)$. Therefore, we assume $\Delta(G) \geq 1$.

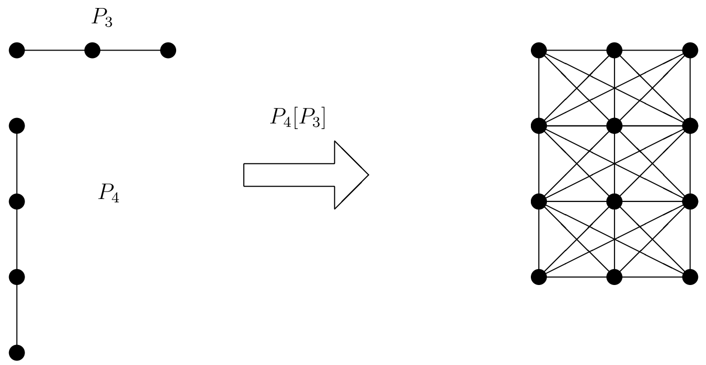

Fig. 1. The composition of two graphs.

The composition of n ($n \geq 2$) graphs $G_1, G_2, \ldots, G_n$, $C_{G_1,G_2,\ldots,G_n}$, is defined recursively by $C_{G_n} = G_n$ and $C_{G_k,G_{k+1},\ldots,G_n} = G_k[C_{G_{k+1},G_{k+2},\ldots,G_n}]$ for $k = n-1, n-2, \ldots, 1$.

In this section, we obtain an upper bound for $\lambda(C_{G_1,G_2,\ldots,G_n})$ in terms of the maximum degrees of $G_1, G_2, \ldots, G_n$, $C_{G_1,G_2,\ldots,G_n}$, using a combinatorial approach.

Theorem 3.1. *Let $G_1, G_2, \ldots, G_n$ be graphs with maximum degrees $\Delta_1, \Delta_2, \ldots, \Delta_n$, respectively, such that $\Delta_1 \geq 1$. Then*

$$\lambda(C_{G_1,G_2,\ldots,G_n}) \leq \beta_2(1 + \Delta_1 + \Delta_1^2) + \alpha - 1,$$

where $\beta_j = |V(G_j)| \times |V(G_{j+1})| \times \cdots \times |V(G_n)|$ for all $j = 1, 2, \ldots, n$, and $\alpha = \sum_{j=2}^{n-1}(\beta_{j+1}\Delta_j) + \Delta_n$.

Proof. Let us apply Algorithm Label to $C_{G_1,G_2,\ldots,G_n}$ and let $x = (i_1, i_2, \ldots, i_n) \in V(C_{G_1,G_2,\ldots,G_n})$ be a vertex with the largest label. Let d be equal to the degree of x in $C_{G_1,G_2,\ldots,G_n}$ and for each $j = 1, 2, \ldots, n$, let us define the following values: d_j equal to the degree of i_j in G_j, $\nu_j = |V(G_j)|$, and $\beta_j = \nu_j\nu_{j+1}\cdots\nu_n$. Let $\beta_{n+1} = 1$. Note from the definition of composition that the number of vertices of $C_{G_j,\ldots,G_n}$ is β_j for all $j = 1, 2, \ldots, n$. Let t be the number of vertices at distance 2 from vertex x in graph $C_{G_1,\ldots,G_n}$.

Observe that graph $C_{G_j,\ldots,G_n}$, $j < n$, can be constructed as follows:

1. Replace each vertex u of G_j with a copy of $C_{G_{j+1},\ldots,G_n}$. Let us denote this copy of $C_{G_{j+1},\ldots,G_n}$ corresponding to vertex u as C^u.
2. For every edge e_{uv} of G_j add an edge between every vertex of C^u and every vertex of C^v.

Therefore, the following sets of vertices contain all the vertices of $C_{G_1,G_2,\ldots,G_n}$ that are at distance 2 from $x = (i_1, i_2, \ldots, i_n)$:

(a) The vertices in the copy C^{i_1} of $C_{G_2,\ldots,G_n}$ corresponding to vertex i_1, with the exception of x and the neighbours of x in C^{i_1}. The number of vertices in C^{i_1} is $\nu_2\nu_3\cdots\nu_n$ and the number of neighbours of x in C^{i_1} is $d - d_1\nu_2\nu_3\cdots\nu_n$ as d is the total number of neighbours of x and $d_1\nu_2\nu_3\cdots\nu_n$ is the number of neighbours of x that do not belong to C^{i_1}.

(b) The vertices not in C^{i_1} at distance 2 from x. There can be at most $d_1(\Delta_1 - 1)\nu_2\nu_3\cdots\nu_n$ such vertices as each neighbour of i_1 in G_1 has at most Δ_1 neighbors.

Hence,

$$\begin{aligned} t &\leq \nu_2\nu_3\cdots\nu_n - (d - d_1\nu_2\cdots\nu_n) - 1 + d_1(\Delta_1 - 1)\nu_2\cdots\nu_n \\ &= \nu_2\cdots\nu_n(1 + d_1\Delta_1 - d_1) - d + d_1\nu_2\cdots\nu_n - 1 \\ &= \beta_2(1 + d_1\Delta_1) - d - 1 \end{aligned}$$

The maximum degree of the graph $C_{G_1,G_2,\ldots,G_n}$ is

$$\begin{aligned} \Delta &= \sum_{j=1}^{n}(\beta_{j+1}\Delta_j) = \beta_2\Delta_1 + \sum_{j=2}^{n}(\beta_{j+1}\Delta_j) \\ &= \beta_2\Delta_1 + \alpha, \quad \text{where } \alpha = \sum_{j=2}^{n}(\beta_{j+1}\Delta_j). \end{aligned} \tag{2}$$

From (1), the bound B for the $L(2, 1)$-labelling number of $C_{G_1,G_2,\ldots,G_n}$ is

$$\begin{aligned} B &= |I_1| + |I_2| \leq d + d + t, \quad \text{by the definition of } I_1, I_2, \text{ and } t \\ &\leq 2d + \beta_2(1 + d_1\Delta_1) - d - 1 \\ &= d + \beta_2(1 + d_1\Delta_1) - 1 \\ &\leq \Delta + \beta_2(1 + d_1\Delta_1) - 1 \\ &= \beta_2\Delta_1 + \alpha + \beta_2(1 + d_1\Delta_1) - 1, \quad \text{by (2)} \\ &= \beta_2(1 + \Delta_1 + d_1\Delta_1) + \alpha - 1. \quad \blacksquare \end{aligned}$$

Corollary 3.2. *Let G, H be graphs with maximum degrees Δ_1, Δ_2 respectively, such that $\Delta_1 \geq 1$. Then*

$$\lambda(G[H]) \leq \beta_2(1 + \Delta_1 + \Delta_1^2) + \alpha - 1 = \nu_2\Delta_1 + \Delta_2 - 1 + \nu_2(1 + \Delta_1^2).$$

In [21], Shiu et al. proved that $\lambda(G[H]) \leq \nu_2\Delta_1 + \Delta_2 + \nu_2\Delta_1^2 + \Delta_2^2$. Because $\nu_2\Delta_1 + \Delta_2 + \nu_2\Delta_1^2 + \Delta_2^2 - (\nu_2\Delta_1 + \Delta_2 - 1 + \nu_2(1 + \Delta_1^2)) = \Delta_2^2 - \nu_2 + 1$, the bound in Corollary 3.2 is better than that of Shiu et al. if $\nu_2 < \Delta_2^2 + 1$.

Lemma 3.3. *Let G_1, G_2 be graphs with maximum degrees Δ_1, Δ_2 and numbers of vertices ν_1, ν_2, respectively, such that $\Delta_1 = 2$ and $\Delta_2 = 0$. Then $\lambda(G_1[G_2]) \leq 5\nu_2 - 1$. In particular, $\lambda(c_5[G_2]) = 5\nu_2 - 1$.*

Proof. Without loss of generality, we can suppose that $G_1 = c_{\nu_1}$, i.e., a cycle with ν_1 ($\nu_1 \geq 3$) vertices. We will give an explicit $(5\nu_2 - 1)$-$L(2, 1)$-labeling l on $G_1[G_2]$ as follows. Let $v_0, \ldots, v_{\nu_1-2}$ be vertices of c_{ν_1} such that v_i is adjacent to v_{i+1}, $0 \leq i \leq \nu_1 - 2$ and v_0 is adjacent to v_{ν_1-1}. Then, consider the following labelling:

Case 1. $\nu_1 \equiv 0 \bmod 3$.

Subcase 1. $i \equiv 0 \bmod 3$. Label each vertex in the copy of G_2 corresponding to v_i in $G_1[G_2]$ successively with $0, 1, \ldots, \nu_2-1$.

Subcase 2. $i \equiv 1 \bmod 3$. Label each vertex in the copy of G_2 corresponding to v_i in $G_1[G_2]$ successively with $\nu_2 + 1, \nu_2 + 2, \ldots, 2\nu_2$.

Subcase 3. $i \equiv 2 \bmod 3$. Label each vertex in the copy of G_2 corresponding to v_i in $G_1[G_2]$ successively with $2\nu_2+2, 2\nu_2+3, \ldots, 3\nu_2 + 1$.

Case 2. $\nu_1 \equiv 1 \bmod 3$. First define l at each vertex in the copy of G_2 corresponding to each vertex $v_0, \ldots, v_{\nu_2-2}$ in $G_1[G_2]$ as:

Subcase 1. $i \equiv 0 \bmod 3$. Label each vertex in the copy of G_2 corresponding to v_i in $G_1[G_2]$ successively with $0, 1, \ldots, \nu_2-1$.

Subcase 2. $i \equiv 1 \bmod 3$. Label each vertex in the copy of G_2 corresponding to v_i in $G_1[G_2]$ successively with $2\nu_2+2, 2\nu_2+3, \ldots, 3\nu_2 + 1$.

Subcase 3. $i \equiv 2 \bmod 3$. Label each vertex in the copy of G_2 corresponding to v_i in $G_1[G_2]$ successively with $\nu_2 + 1, \nu_2 + 2, \ldots, 2\nu_2$.

Finally, define l at each vertex in the copy of G_2 corresponding to v_{ν_2-1} in $G_1[G_2]$ as follows: label each vertex in this copy of G_2 successively with $3\nu_2 + 2, 3\nu_2 + 3, \ldots, 4\nu_2 + 1$.

Case 3. $\nu_1 \equiv 2 \bmod 3$. First define l at each vertex in the copy of G_2 corresponding to each vertex $v_0, \ldots, v_{\nu_2-3}$ in $G_1[G_2]$ as:

Subcase 1. $i \equiv 0 \bmod 3$. Label each vertex in the copy of G_2 corresponding to v_i in $G_1[G_2]$ successively with $0, 1, \ldots, \nu_2-1$.

Subcase 2. $i \equiv 1 \bmod 3$. First label each vertex except the last one in the copy of G_2 corresponding to v_i in $G_1[G_2]$ successively with $\nu_2 + 1, \nu_2 + 2, \ldots, 2\nu_2 - 1$ and then label the last vertex with $4\nu_2$.

Subcase 3. $i \equiv 2 \bmod 3$. Label each vertex in the copy of G_2 corresponding to v_i in $G_1[G_2]$ successively with $2\nu_2+1, 2\nu_2+2, \ldots, 3\nu_2$.

Then define l at each vertex in the copy of G_2 corresponding to v_{ν_2-2} in $G_1[G_2]$ as follows: first label each vertex except the last one in this copy of G_2 successively with $4\nu_2 + 1, 4\nu_2 + 2, \ldots, 5\nu_2 - 1$ and then label the last vertex with ν_2; finally define l at each vertex in the copy of G_2 corresponding to v_{ν_2-1} in $G_1[G_2]$ as follows: first label each vertex except the last one in this copy of G_2 successively with $3\nu_2 + 1, 3\nu_2 + 2, \ldots, 4\nu_2 - 1$ and then label the last vertex with $2\nu_2$.

It is easy to verify that the above labeling scheme is feasible. Then $\lambda(G_1[G_2]) \leq 5\nu_2 - 1$.

Note that since c_5 is a diameter 2 graph, then $c_5[G_2]$ is also a diameter 2 graph; therefore all vertices of $c_5[G_2]$ must be assigned different labels. Thus, $\lambda(c_5[G_2]) \geq 5\nu_2 - 1$. But $\lambda(c_5[G_2]) \leq 5\nu_2 - 1$. Then $\lambda(c_5[G_2]) = 5\nu_2 - 1$. That is, the above labelling scheme is optimal for $c_5[G_2]$. ■

Lemma 3.4. *Let G_1, G_2 be graphs with maximum degrees Δ_1, Δ_2 and numbers of vertices ν_1, ν_2, respectively, such that $\Delta_1 \geq 1$ and $\Delta_2 = 0$. Then $\lambda(C_{G_1,G_2}) \leq \Delta^2 - \Delta$ where Δ is the maximum degree of C_{G_1,G_2}, with the only exceptions that $\lambda(C_{G_1,G_2}) \leq \Delta^2 + \Delta$ when $\Delta_1 \geq 3$ and $\nu_2 = 1$ or $\lambda(C_{G_1,G_2}) = \Delta^2$ when C_{G_1,G_2} consists of copies of c_4.*

Proof. Because $\Delta_2 = 0$, the number of vertices at distance 1 from x is at most $\nu_2\Delta_1$ and the number of vertices at distance 2 from x is at most $\nu_2\Delta_1(\Delta_1 - 1) + \nu_2 - 1$. Hence $|I_1| \leq \nu_2\Delta_1$, $|I_2| \leq \nu_2\Delta_1 + \nu_2\Delta_1(\Delta_1 - 1) + \nu_2 - 1$. Then $B = |I_1| + |I_2| \leq \nu_2\Delta_1 + \nu_2\Delta_1 + \nu_2\Delta_1(\Delta_1 - 1) + \nu_2 - 1 = \nu_2\Delta_1^2 + \nu_2\Delta_1 + \nu_2 - 1$. We need to consider three cases.

Case 1. $\Delta_1 \geq 3$.

Subcase 1. $\nu_2 = 1$. Then $C_{G_1,G_2} = G_1$. In this case, C_{G_1,G_2} is the general graph G_1 with maximum degree $\Delta_1 \geq 3$.

Subcase 2. $\nu_2 \geq 2$. We have $(\nu_2\Delta_1)^2 - \nu_2\Delta_1 - (\nu_2\Delta_1^2 + \nu_2\Delta_1 + \nu_2 - 1) = \nu_2((\nu_2-1)\Delta_1^2 - 2\Delta_1 - 1) + 1 \geq \nu_2(9\nu_2 - 16) + 1 = 9\nu_2^2 - 16\nu_2 + 1 \geq 2\nu_2 + 1$. Hence $B \leq (\nu_2\Delta_1)^2 - \nu_2\Delta_1 - (2\nu_2 + 1) = \Delta^2 - \Delta - (2\nu_2 + 1)$.

Case 2. $\Delta_1 = 2$. By Lemma 3.3, we have $\lambda(G_1[G_2]) \leq 5\nu_2 - 1$.

But $(\nu_2\Delta_1)^2 - \nu_2\Delta_1 - (5\nu_2 - 1) = 4\nu_2^2 - 7\nu_2 + 1 \geq 3$, so $\lambda(G_1[G_2]) \leq (\nu_2\Delta_1)^2 - \nu_2\Delta_1 - 3 = \Delta^2 - \Delta - 3$.

Case 3. $\Delta_1 = 1$. Then $\lambda(G_1[G_2]) = 2\nu_2$.

If $\nu_2 \geq 3$, then $(\nu_2\Delta_1)^2 - \nu_2\Delta_1 - 2\nu_2 = \nu_2^2 - 3\nu_2 \geq 0$. Hence $\lambda(G_1[G_2]) \leq \Delta^2 - \Delta$.

If $\nu_2 = 2$, then $G_1[G_2]$ consists of copies of c_4. Hence $\lambda(G_1[G_2]) = 4 \leq \Delta^2$. ■

Lemma 3.5. *Let $G_1, G_2, \ldots, G_n$ be graphs with maximum degrees $\Delta_1, \Delta_2, \ldots, \Delta_n$, respectively, such that $\Delta_1 \geq 1$. Then $\lambda(C_{G_1,G_2,\ldots,G_n}) \leq \Delta^2 - \Delta$, where Δ is the maximum degree of $C_{G_1,G_2,\ldots,G_n}$, with the only exceptions that $\lambda(C_{G_1,G_2,\ldots,G_n}) \leq \Delta^2 + \Delta$ when $\nu_2 = \nu_3 = \cdots = \nu_n = 1$ or $\lambda(C_{G_1,G_2,\ldots,G_n}) = \Delta^2$ where $C_{G_1,G_2,\ldots,G_n}$ consists of copies of c_4.*

Proof. From Theorem 3.1, $\lambda(C_{G_1,G_2,\ldots,G_n}) \leq \beta_2(1 + \Delta_1 + \Delta_1^2) + \alpha - 1$ so we just need to show that this bound is at most $\Delta^2 - \Delta$, except when $\nu_2 = \nu_3 = \cdots = \nu_n = 1$ or $C_{G_1,G_2,\ldots,G_n}$ consists of copies of c_4.

$$\begin{aligned}
&\Delta^2 - \Delta - (\beta_2(1 + \Delta_1 + \Delta_1^2) + \alpha - 1)\\
&\quad = (\beta_2\Delta_1 + \alpha)^2 - (\beta_2\Delta_1 + \alpha) - (\beta_2(1 + \Delta_1 + \Delta_1^2) + \alpha - 1), \quad \text{from (2)}\\
&\quad = (\beta_2^2 - \beta_2)\Delta_1^2 + 2\beta_2\Delta_1(\alpha - 1) + \alpha^2 - 2\alpha - \beta_2 + 1
\end{aligned}$$

We now need to consider three cases.

Case 1. $\alpha = \sum_{j=2}^{n}(\beta_{j+1}\Delta_j) = 0$. Then $\Delta_j = 0, j = 2, \ldots, n$. By Lemma 3.3, the conclusion holds.

Case 2. $\alpha = \sum_{j=2}^{n}(\beta_{j+1}\Delta_j) = 1$. Then

Subcase 1. $\beta_2 = 1$. Since $\beta_2 = \nu_2\nu_3\cdots\nu_n = 1$, then $\nu_2 = \nu_3 = \cdots = \nu_n = 1$. Hence $C_{G_1,G_2,\ldots,G_n} = G_1$. In this case, $C_{G_1,G_2,\ldots,G_n}$ is the general graph G_1 with maximum degree $\Delta_1 \geq 1$.

Subcase 2. $\beta_2 \geq 2$. We have $\Delta^2 - \Delta - (\beta_2(1+\Delta_1+\Delta_1^2)+\alpha-1) = (\beta_2^2-\beta_2)\Delta_1^2+2\beta_2\Delta_1(\alpha-1)+\alpha^2-2\alpha-\beta_2+1 = (\beta_2^2 - \beta_2)\Delta_1^2 - \beta_2 = \beta_2((\beta_2 - 1)\Delta_1^2 - 1) \geq \beta_2(\Delta_1^2 - 1) \geq 0$. Then the conclusion holds.

Case 3. $\alpha = \sum_{j=2}^{n}(\beta_{j+1}\Delta_j) \geq 2$. Then $\Delta^2-\Delta-(\beta_2(1+\Delta_1+\Delta_1^2)+\alpha-1) = (\beta_2^2-\beta_2)\Delta_1^2+2\beta_2\Delta_1(\alpha-1)+\alpha^2-2\alpha-\beta_2+1 \geq (\beta_2^2 - \beta_2)\Delta_1^2 + \beta_2(2\Delta_1 - 1) + 1 \geq \beta_2^2 + 1$ (since $\beta_2 \geq 2$ and $\Delta_1 \geq 1$). Then the conclusion holds. ■

By the proof of Lemma 3.4, the bound in Theorem 3.1 is much smaller than $\Delta^2 - \Delta$ if $\alpha \geq 2$ or $\alpha = 1$ and $\Delta_1 \geq 2$.

4. Correction to the proof in [19] for the composition of two graphs

Theorem 4.3 in [19] states a bound for $\lambda(C_{G_1,G_2})$ by establishing a lower bound on ε, the number of edges of the subgraph F induced by the neighbors of x labelled with the largest label by algorithm Label. Unfortunately, the proof of the theorem given in [19] is not totally correct because if vertex x is isolated in G_2, then the lower bound for ε will not hold and therefore the upper bound for $\lambda(G_1[G_2])$ cannot be established by this method, but if vertex x is not isolated in G_2, then the lower bound for ε will still hold and therefore the proof is still correct.

In this section, we fix the proof of that theorem.

Theorem 4.1 (*[19]*). *Let the maximum degree of $G_1[G_2]$ be Δ. Then $\lambda(G_1[G_2]) \leq \Delta^2 + \Delta - 2\nu_2\Delta_1$ or $\lambda(G_1[G_2]) \leq \Delta^2 - \Delta$, with the only exceptions that $\lambda(C_{G_1,G_2}) \leq \Delta^2 + \Delta$ when $\Delta_1 \geq 3$ and $\nu_2 = 1$ or $\lambda(G_1[G_2]) = \Delta^2$ when $G_1[G_2]$ consists of copies of c_4.*

Proof. Again, we apply Algorithm Label to obtain a $L(2,1)$-labeling with maximum label k on the graph $G_1[G_2]$. Let $x \in V(G_1[G_2])$ be labeled with k. We only consider the case when the degree of x in G_2 is zero.

Case 1. $\Delta_2 > 0$. Because x is isolated in G_2, the number of vertices at distance 1 from x is at most $\nu_2\Delta_1$ and the number of vertices at distance 2 from x is at most $\nu_2\Delta_1(\Delta_1 - 1) + \nu_2 - 1$. Hence $|I_1| \leq \nu_2\Delta_1$, $|I_2| \leq \nu_2\Delta_1 + \nu_2\Delta_1(\Delta_1 - 1) + \nu_2 - 1$. Then $B = |I_1| + |I_2| \leq \nu_2\Delta_1 + \nu_2\Delta_1 + \nu_2\Delta_1(\Delta_1 - 1) + \nu_2 - 1 = \nu_2\Delta_1^2 + \nu_2\Delta_1 + \nu_2 - 1$.

Since $\Delta_2 > 0$ and x is isolated in G_2, $\nu_2 \geq 3$. Note that $\Delta_1 \geq 1$ and $\Delta_2 \geq 1$; then $(\nu_2\Delta_1 + \Delta_2)^2 - (\nu_2\Delta_1 + \Delta_2) - (\nu_2\Delta_1^2 + \nu_2\Delta_1 + \nu_2 - 1) = \nu_2((\nu_2 - 1)\Delta_1^2 + 2\Delta_1(\Delta_2 - 1)) + \Delta_2(\Delta_2 - 1) + \nu_2 - 1 \geq \nu_2(\nu_2 - 1)\Delta_1^2 + \nu_2 - 1 \geq \nu_2(\nu_2 - 1) + \nu_2 - 1 = \nu_2^2 - 1 \geq 8$. Hence $B \leq (\nu_2\Delta_1 + \Delta_2)^2 - (\nu_2\Delta_1 + \Delta_2) - (\nu_2^2 - 1) = \Delta^2 - \Delta - (\nu_2^2 - 1) \leq \Delta^2 - \Delta - 8$.

Case 2. $\Delta_2 = 0$. The proof is the same as for Lemma 3.3. ■

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